▷ **1.11** Let $N = \{1, 2, \ldots, 100\}$, and $A \subseteq N$ with $|A| = 55$. Show that A contains two numbers with difference 9. Is this also true for $|A| = 54$?

1.2 Subsets and Binomial Coefficients

Let N be an n-set. We have already introduced the *binomial coefficient* $\binom{n}{k}$ as the number of k-subsets of N. To derive a formula for $\binom{n}{k}$ we look first at words of length k with symbols from N.

Definition. A *k-permutation* of N is a k-word over N all of whose entries are distinct.

For example, 1235 and 5614 are 4-permutations of $\{1, 2, \ldots, 6\}$. The number of k-permutations is quickly computed. We have n possibilities for the first letter. Once we have chosen the first entry, there are $n-1$ possible choices for the second entry, and so on. The product rule thus gives the following result:

The number of k-permutations of an n-set equals $n(n-1)\cdots(n-k+1)$ $(n, k \geq 0)$.

For $k = n$ we obtain, in particular, $n! = n(n-1)\cdots 2 \cdot 1$ for the number of n-permutations, i.e., of ordinary permutations of N. As usual, we set $0! = 1$.

The expressions $n(n-1)\cdots(n-k+1)$ appear so frequently in enumeration problems that we give them a special name:

> $n^{\underline{k}} := n(n-1)\cdots(n-k+1)$ are the *falling factorials* of length k, with $n^{\underline{0}} = 1$ $(n \in \mathbb{Z},\ k \in \mathbb{N}_0)$.

Similarly,

> $n^{\overline{k}} := n(n+1)\cdots(n+k-1)$ are the *rising factorials* of length k, with $n^{\overline{0}} = 1$ $(n \in \mathbb{Z},\ k \in \mathbb{N}_0)$.

Now, every k-permutation consists of a unique k-subset of N. Since every k-subset can be permuted in $k!$ ways to produce a k-permutation, counting in two ways gives $k!\binom{n}{k} = n^{\underline{k}}$, hence

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!} \quad (n, k \geq 0)\,, \tag{1}$$

where, of course, $\binom{n}{k} = 0$ for $n < k$.

Another way to write (1) is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (n \geq k \geq 0), \tag{2}$$

from which $\binom{n}{k} = \binom{n}{n-k}$ results.

Identities and formulas involving binomial coefficients fill whole books; Chapter 5 of Graham–Knuth–Patashnik gives a comprehensive survey. Let us just collect the most important facts.

Pascal Recurrence.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad \binom{n}{0} = 1 \quad (n, k \geq 0). \tag{3}$$

We have already proved this recurrence in Section 1.1; it also follows immediately from (1).

Now we make an important observation, the so-called *polynomial method*. The polynomials

$$x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1), x^{\overline{k}} = x(x+1)(x+2)\cdots(x+k-1)$$

over $\mathbb{C}$ (or any field of characteristic 0) are again called the falling resp. rising factorials, where $x^{\underline{0}} = x^{\overline{0}} = 1$. Consider the polynomials

$$\frac{x^{\underline{k}}}{k!} \quad \text{and} \quad \frac{(x-1)^{\underline{k-1}}}{(k-1)!} + \frac{(x-1)^{\underline{k}}}{k!}.$$

Both have degree k, and we know that two polynomials of degree k that agree in more than k values are identical. But in our case they even agree for *infinitely* many values, namely for all non-negative integers, and so we obtain the *polynomial* identity

$$\frac{x^{\underline{k}}}{k!} = \frac{(x-1)^{\underline{k-1}}}{(k-1)!} + \frac{(x-1)^{\underline{k}}}{k!} \quad (k \geq 1). \tag{4}$$

Thus, if we set $\binom{c}{k} = \frac{c^{\underline{k}}}{k!} = \frac{c(c-1)\cdots(c-k+1)}{k!}$ for arbitrary $c \in \mathbb{C}$ ($k \geq 0$), then Pascal's recurrence holds for $\binom{c}{k}$. In fact, it is convenient to extend the definition to negative integers k, setting

$$\binom{c}{k} = \begin{cases} \frac{c^{\underline{k}}}{k!} & (k \geq 0) \\ 0 & (k < 0)\,. \end{cases}$$

Pascal's recurrence holds then in general, since for $k < 0$ both sides are 0:

$$\binom{c}{k} = \binom{c-1}{k-1} + \binom{c-1}{k} \qquad (c \in \mathbb{C}, k \in \mathbb{Z})\,. \tag{5}$$

As an example, $\binom{-1}{n} = \frac{(-1)(-2)\cdots(-n)}{n!} = (-1)^n$.

Here is another useful polynomial identity. From

$$(-x)^{\underline{k}} = (-x)(-x-1)\cdots(-x-k+1) = (-1)^k x(x+1)\cdots(x+k-1)$$

we get

$$(-x)^{\underline{k}} = (-1)^k x^{\overline{k}}, \quad (-x)^{\overline{k}} = (-1)^k x^{\underline{k}}. \tag{6}$$

With $x^{\overline{k}} = (x+k-1)^{\underline{k}}$ this gives

$$\binom{-c}{k} = (-1)^k \binom{c+k-1}{k}, \quad (-1)^k \binom{c}{k} = \binom{k-c-1}{k}. \tag{7}$$

Equation (6) is called the *reciprocity law* between the falling and rising factorials.

The recurrence (3) gives the Pascal matrix $P = \left(\binom{n}{k}\right)$ with n as row index and k as column index. P is a lower triangular matrix with 1's on the main diagonal. The table shows the first rows and columns, where the 0's are omitted.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1							

There are many beautiful and sometimes mysterious relations in the Pascal matrix to be discovered. Let us note a few formulas that we will need time and again. First, it is clear that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$, since we are counting all subsets of an n-set. Consider the column-sum of index k down to row n, i.e., $\sum_{i=0}^{n}\binom{i}{k}$. By classifying the $(k+1)$-subsets of $\{1, 2, \ldots, n+1\}$ according to the last element $i+1$ $(0 \le i \le n)$ we obtain

$$\sum_{i=0}^{n}\binom{i}{k} = \binom{n+1}{k+1}. \tag{8}$$

Let us next look at the down diagonal from left to right, starting with row m and column 0. That is, we want to sum $\sum_{i=0}^{n}\binom{m+i}{i}$. In the table above, the diagonal with $m = 3, n = 3$ is marked, summing to $35 = \binom{7}{3}$. Writing $\sum_{i=0}^{n}\binom{m+i}{i} = \sum_{i=0}^{n}\binom{m+i}{m} = \sum_{k=0}^{m+n}\binom{k}{m}$, this is just a sum like that in (8), and we obtain

$$\sum_{i=0}^{n}\binom{m+i}{i} = \binom{m+n+1}{n}. \tag{9}$$

Note that (9) holds in general for $m \in \mathbb{C}$.

From the reciprocity law (7) we may deduce another remarkable formula. Consider the *alternating* partial sums in row 7: $1, 1-7 = -6, 1-7+21 = 15, -20, 15, -6, 1, 0$. We note that these are precisely the binomial coefficients immediately above, with alternating sign. Let us prove this in general; (7) and (9) imply

$$\sum_{k=0}^{m}(-1)^k\binom{n}{k} = \sum_{k=0}^{m}\binom{k-n-1}{k} = \binom{m-n}{m} = (-1)^m\binom{n-1}{m}. \tag{10}$$

The reader may wonder whether there is also a simple formula for the partial sums $\sum_{k=0}^{m}\binom{n}{k}$ without signs. We will address this question of when a “closed” formula exists in Chapter 4 (and the answer for this particular case will be no).

Next, we note an extremely useful identity that follows immediately from (2); you are asked in the exercises to provide a combinatorial argument:

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k} \quad (n, m, k \in \mathbb{N}_0). \tag{11}$$

Binomial Theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} . \tag{12}$$

Expand the left-hand side, and classify according to the number of x's taken from the factors. The formula is an immediate consequence.

For $y = 1$ respectively $y = -1$ we obtain

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \ (x-1)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^k , \tag{13}$$

and hence for $x = 1$, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ and

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \delta_{n,0} , \tag{14}$$

where $\delta_{i,j}$ is the Kronecker symbol

$$\delta_{i,j} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

This last formula will be the basis for the inclusion–exclusion principle in Chapter 5. We may prove (14) also by the bijection principle. Let N be an n-set, and set $S_0 = \{A \subseteq N : |A| \text{ even}\}$, $S_1 = \{A \subseteq N : |A| \text{ odd}\}$. Formula (14) is then equivalent to $|S_0| = |S_1|$ for $n \geq 1$. To see this, pick $a \in N$ and define $\phi : S_0 \to S_1$ by

$$\phi(A) = \begin{cases} A \cup a & \text{if } a \notin A, \\ A \smallsetminus a & \text{if } a \in A. \end{cases}$$

This is a desired bijection.

Vandermonde Identity.

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} \quad (n \in \mathbb{N}_0) . \tag{15}$$

Once again the polynomial method applies. Let R and S be disjoint sets with $|R| = r$ and $|S| = s$. The number of n-subsets of $R \cup S$ is $\binom{r+s}{n}$. On the other hand, any such set arises by combining a k-subset of R with an $(n-k)$-subset of S. Classifying the n-subsets A according to $|A \cap R| = k$ yields

$$\binom{r+s}{n} = \sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} \quad \text{for all } \ r, s \in \mathbb{N}_0 \,.$$

The polynomial method completes the proof.

Example. We have $\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$.

Multiplying both sides of (15) by $n!$ we arrive at a "binomial" theorem for the falling factorials:

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}} \tag{16}$$

and the reciprocity law (6) gives the analogous statement for the rising factorials:

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}} \,. \tag{17}$$

Multisets.

In a set all elements are distinct, in a *multiset* we drop this requirement. For example, $M = \{1, 1, 2, 2, 3\}$ is a multiset over $\{1, 2, 3\}$ of size 5, where 1 and 2 appear with multiplicity 2. Thus the size of a multiset is the number of elements counted with their multiplicities. The following formula shows the importance of rising factorials:

The number of k-multisets of an n-set is

$$\frac{n^{\overline{k}}}{k!} = \frac{n(n+1)\cdots(n+k-1)}{k!} = \binom{n+k-1}{k} \,. \tag{18}$$

Just as a k-subset A of $\{1, 2, \ldots, n\}$ can be interpreted as a *monotone* k-word $A = \{1 \le a_1 < a_2 < \cdots < a_k \le n\}$, a k-multiset is a monotone k-word *with repetitions* $\{1 \le a_1 \le \cdots \le a_k \le n\}$. This interpretation immediately leads to a proof of (18) by the bijection rule. The map $\phi : A = \{a_1 \le a_2 \le \cdots \le a_k\} \longrightarrow A' = \{1 \le a_1 < a_2 + 1 < a_3 + 2 < \cdots < a_k + k - 1 \le n + k - 1\}$ is clearly a bijection, and (18) follows.

Multinomial Theorem.

$$(x_1 + \cdots + x_m)^n = \sum_{(k_1,\cdots,k_m)} \binom{n}{k_1 \dots k_m} x_1^{k_1} \cdots x_m^{k_m}\,, \tag{19}$$

where

$$\binom{n}{k_1 \dots k_m} = \frac{n!}{k_1! \cdots k_m!}, \quad \sum_{i=1}^{m} k_i = n\,, \tag{20}$$

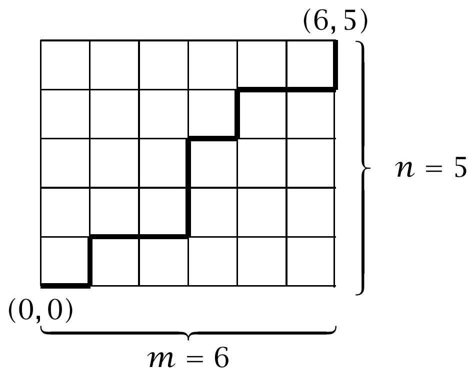
is the *multinomial coefficient.*

The proof is similar to that of the binomial theorem. Expanding the left-hand side we pick x_1 out of k_1 factors; this can be done in $\binom{n}{k_1} = \frac{n!}{k_1!(n-k_1)!}$ ways. Out of the remaining $n - k_1$ factors we choose x_2 from k_2 factors in $\binom{n-k_1}{k_2} = \frac{(n-k_1)!}{k_2!(n-k_1-k_2)!}$ ways, and so on.

A useful interpretation of the multinomial coefficients is the following. The ordinary binomial coefficient $\binom{n}{k}$ counts the number of n-words over $\{x, y\}$ with exactly k x's and $n - k$ y's. Similarly, the multinomial coefficient $\binom{n}{k_1 \dots k_m}$ is the number of n-words over an alphabet $\{x_1, \dots, x_m\}$ in which x_i appears exactly k_i times.

Lattice Paths.

Finally, we discuss an important and pleasing way to look at binomial coefficients. Consider the $(m \times n)$-lattice of integral points in $\mathbb{Z}^2$, e.g., $m = 6$, $n = 5$ as in the figure,



and look at all lattice paths starting at $(0,0)$, terminating at (m,n), with steps one to the right or one upward. We will call the horizontal steps $(1,0)$-steps since the x-coordinate is increased by 1, and similarly, we call the vertical steps $(0,1)$-steps. Let $L(m,n)$ be the number of these lattice paths. The initial conditions are $L(m,0) = L(0,n) = 1$, and classification according to the first step immediately gives

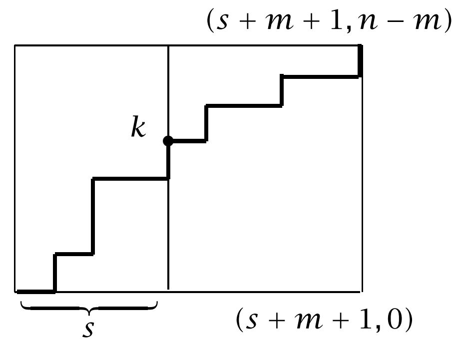
$$L(m,n) = L(m-1,n) + L(m,n-1)\,.$$

This is precisely Pascal's recurrence for $\binom{m+n}{m}$, and we conclude that

$$L(m,n) = \binom{m+n}{m}\,. \tag{21}$$

Another quick way to see this is by encoding the paths. We assign the symbol E(ast) to a $(1,0)$-step and N(orth) to a $(0,1)$-step. The lattice paths correspond then bijectively to $(m+n)$-words over $\{E,N\}$ with precisely m E's, and this is $\binom{m+n}{m}$. In the example above, the encoding is given by ENEENNENEEN. The lattice path interpretation allows easy and elegant proofs of many identities involving binomial coefficients.

Example. Consider the following variant of the Vandermonde identity: $\sum_{k=0}^{n}\binom{s+k}{k}\binom{n-k}{m} = \binom{s+n+1}{s+m+1}$ $(s,m,n \in \mathbb{N}_0)$. For $n < m$, both sides are 0, so assume $n \ge m$, and look at the $(s+m+1)\times(n-m)$-lattice. The number of paths is $\binom{s+n+1}{s+m+1}$. Now we classify the paths according to the *highest* coordinate $y = k$ where they touch the vertical line $x = s$.



Then the next step is a $(1,0)$-step, and the sum and product rules give

$$\binom{s+n+1}{s+m+1} = \sum_{k=0}^{n-m} \binom{s+k}{k}\binom{m+(n-m-k)}{m}$$
$$= \sum_{k=0}^{n} \binom{s+k}{k}\binom{n-k}{m}.$$

Exercises

1.12 Prove $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ $(n \geq m \geq k \geq 0)$ by counting pairs of sets (A, B) in two ways, and deduce $\sum_{k=0}^{m}\binom{n}{k}\binom{n-k}{m-k} = 2^m\binom{n}{m}$.

▷ **1.13** Use the previous exercise to show that $\binom{2n}{2k}\binom{2n-2k}{n-k}\binom{2k}{k} = \binom{2n}{n}\binom{n}{k}^2$ for $n \geq k \geq 0$.

1.14 Show that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, $\binom{n}{k} = \frac{k+1}{n-k}\binom{n}{k+1}$, and use this to verify the *unimodal* property for the sequence $\binom{n}{k}$, $0 \leq k \leq n$: $\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n}$.

1.15 Show that that the sum of right–left diagonals in the Pascal matrix ending at $(n, 0)$ is the Fibonacci number F_{n+1}, i.e., $F_{n+1} = \sum_{k\geq 0}\binom{n-k}{k}$.

1.16 Show that $r^{\underline{k}}(r - \frac{1}{2})^{\underline{k}} = \frac{(2r)^{\underline{2k}}}{2^{2k}}$ $(r \in \mathbb{C}, k \in \mathbb{N}_0)$, and deduce $\binom{-1/2}{n} = (-\frac{1}{4})^n\binom{2n}{n}$, $\binom{-3/2}{n} = (-\frac{1}{4})^n(2n+1)\binom{2n}{n}$.

▷ **1.17** Show that the multinomial coefficient $\binom{n}{n_1 \ldots n_k}$ assumes for fixed n and k its maximum in the "middle," where $|n_i - n_j| \leq 1$ for all i, j. Prove in particular that $\binom{n}{n_1 n_2 n_3} \leq \frac{3^n}{n+1}$ $(n \geq 1)$.

1.18 Prove the identities (8) and (9) by counting lattice paths.

* * *

▷ **1.19** The Pascal matrix (slightly shifted) gives a curious prime number test. Index rows and columns as usual by $0, 1, 2, \ldots$. In row n we insert the $n+1$ binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, but shifted to the columns $2n, \ldots, 3n$. In addition, we draw a circle around each of these numbers that is a multiple of n, as in the table.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1			1	1									
2					1	2	1						
3							1	3	3	1			
4									1	4	6	4	1

Show that k is a prime number if and only if all elements in column k are circled. Hint: k even is easy, and for odd k the element in position (n,k) is $\binom{n}{k-2n}$.

1.20 Let $a_n = \frac{1}{\binom{n}{0}} + \frac{1}{\binom{n}{1}} + \cdots + \frac{1}{\binom{n}{n}}$. Show that $a_n = \frac{n+1}{2n}a_{n-1} + 1$ and compute $\lim_{n\to\infty} a_n$ (if the limit exists). Hint: $a_n > 2 + \frac{2}{n}$ and $a_{n+1} < a_n$ for $n \geq 4$.

▷ **1.21** Consider $(m+n)$-words with exactly m 1's and n 0's. Count the number of these words with exactly k runs, where a run is a maximal subsequence of consecutive 1's. Example: 1011100110 has 3 runs.

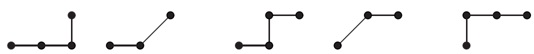
1.22 Prove the following variants of Vandermonde's identity algebraically (manipulating binomial coefficients) and by counting lattice paths.

a. $\sum\limits_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$, b. $\sum\limits_k \binom{r}{m+k}\binom{s}{n+k} = \binom{r+s}{r-m+n}$,

1.23 Give a combinatorial argument for the identity

$$\sum_k \binom{2r}{2k-1}\binom{k-1}{s-1} = 2^{2r-2s+1}\binom{2r-s}{s-1}, r,s \in \mathbb{N}_0 .$$

▷ **1.24** Consider the $(m \times n)$-lattice in $\mathbb{Z}^2$. A *Delannoy path* from $(0,0)$ to (m,n) uses steps $(1,0),(0,1)$ and diagonal steps $(1,1)$ from (x,y) to $(x+1,y+1)$. The number of these paths is the *Delannoy number* $D_{m,n}$. Example for $D_{2,1} = 5$:

 .

Prove that $D_{m,n} = \sum_k \binom{m}{k}\binom{n+k}{m}$. Hint: Classify the paths according to the number of diagonal steps.

1.25 Prove the identity $\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}$, $m,n \in \mathbb{N}_0$. Hint: Write $\binom{r+k}{m+n} = \sum_i \binom{r}{m+n-i}\binom{k}{i}$, and apply (11).

1.26 Prove that $\frac{2^{2n}}{2\sqrt{n}} \leq \binom{2n}{n} \leq \frac{2^{2n}}{\sqrt{n}}$ for $n \geq 1$. Hint: For the upper bound prove the stronger result $\binom{2n}{n} \leq 2^{2n} \,/\, (1+\frac{1}{n})\sqrt{n}$.